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A CHARACTERIZATION OF THE $MM^\star$-POSITION OF A CONVEX BODY IN TERMS OF COVARIANCE MATRICES

BY

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ABSTRACT

We characterize the position of a convex body K such that it minimizes $M(TK)M^\star(TK)$ (the $MM^\star$-position) in terms of properties of the measures $\|\cdot\|_K d\sigma(\cdot)$ and $\|\cdot\|_{K^\circ} d\sigma(\cdot)$, answering a question posed by A. Giannopoulos and V. Milman. The techniques used allow us to study other extremal problems in the context of dual Brunn–Minkowski theory.

1. Introduction and notation

In [GM] A. Giannopoulos and V. Milman characterized extremal positions of convex bodies by the existence of some isotropic measures associated to them and they showed that there are deep relations between the solutions of different extremal problems involving convex bodies and the existence of some measures with isotropic type properties. Following these ideas, the authors in [BR] considered similar problems for extremal positions of convex bodies but in the framework of the dual Brunn–Minkowski theory, and they realized that there also strong relations exist between the solutions of extremal problems and properties of isotropic type of some Borel measures. The aim of this work is to study around these ideas and answer a question posed by A. Giannopoulos and V. Milman in [GM] about

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positions of convex bodies minimizing $M(TK)M^\star(TK)$. If $K \subseteq \mathbb{R}^n$ is a convex body, $M(K)$ is defined by

$$M(K) = \frac{1}{n|D_n|}\int_{S^{n-1}} \|x\|_K d\sigma(x),$$

where D_n denotes the the Euclidean ball in $\mathbb{R}^n$, $|\cdot|$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and $\|\cdot\|_K$ is the gauge of K. In the same way we define $M^\star(K)$ by $M^\star(K) = M(K^\circ)$, where K° is the polar of K given by

$$K^\circ = \{x \in \mathbb{R}^n\colon \langle x, y\rangle \leq 1 \ \forall y \in K\}.$$

It is a central topic in the context of local theory of Banach spaces to give upper estimates for $\min\{M(TK)M^\star(TK)\colon T \in GL(n)\}$, since they have many remarkable applications. In this approach, T. Figiel, N. Tomczak-Jaegermann (see [FT]) and G. Pisier (see [Pi]) proved that for every centrally symmetric convex body $K \subseteq \mathbb{R}^n$ there exists a position TK (i.e., a regular transformation $T \in GL(n)$), called $MM^\star$**-position**, such that

$$M(TK)M^\star(TK) \leq C \log n,$$

for some absolute constant $C > 0$. This upper estimate is known as the $MM^\star$-**estimate** of K. For a general convex body $K \subseteq \mathbb{R}^n$ an $MM^\star$-estimate was given by M. Rudelson (see [R]), who proved that there exists an affine position $t + TK$ of K (involving the Santaló point) such that

$$M(t + TK)M^\star(t + TK) \leq Cn^{1/3}\log^a(n).$$

In [GM], A. Giannopoulos and V. Milman tried to characterize when a convex body $K \subseteq \mathbb{R}^n$ verifies that

$$M(K)M^\star(K) = \min\{M(TK)M^\star(TK)\colon T \in GL(n)\} \tag{1.1}$$

in terms of the probability Borel measures on S^{n-1} defined by

$$d\mu_K(u) = \frac{\|u\|_K}{\int_{S^{n-1}}\|v\|_K d\sigma(v)} d\sigma(u),$$

where $d\sigma(\cdot)$ denotes the $(n-1)$-dimensional Hausdorff measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Actually, they proved that a necessary condition for a symmetric convex body K to verify (1.1) is that $d\mu_K(\cdot)$ and $d\mu_{K^\circ}(\cdot)$ have the same covariance matrix. The main goal of this paper is to show that this kind of conditions are also sufficient conditions and we prove the following result:

THEOREM 1.1: *A symmetric convex body K in $\mathbb{R}^n$ having the origin in its interior is in $MM^\star$-position if and only if the probabilities μ_K and μ_{K^0} have the same covariance matrices.*

The techniques we use allow us to study this problem in a more general framework: the dual Brunn–Minkowski theory, obtaining results for convex bodies not centrally symmetric. If K is a convex body having the origin in its interior, the i-th dual quermassintegral of K (see [L]) is given by

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)d\sigma(u),$$

where ρ_K is the radial function of K defined by $\rho_K(x) = \max\{\lambda > 0\colon \lambda x \in K\}$. Note that $\rho_K(x) = 1/\|x\|_K = 1/h_{K^\circ}(x)$, where $h_K(\cdot)$ is the support function of K. Since

$$M(K) = \frac{1}{|D_n|}\tilde{W}_{n+1}(K),$$

we can extend the extremal problem (1.1) to the context of dual mixed volumes as

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ)\colon T \in GL(n)\}, \tag{1.2}$$

for all $i \in \mathbb{R}$. In fact, it is known that

$$\lim_{\substack{T\in SL(n)\\ \|T\|\to\infty}} \tilde{W}_i(TK) = \begin{cases} 0 & \text{if } i \in (0,n) \\ +\infty & \text{if } i \in (-\infty,0)\cup(n,+\infty) \end{cases}$$

(see [BR], lemma 2.5), which shows that

$$\lim_{\substack{T\in GL(n)\\ \|T\|\to\infty}} \tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) = \begin{cases} 0 & \text{if } i \in (0,n), \\ +\infty & \text{if } i \in (-\infty,0)\cup(n,+\infty), \end{cases}$$

and therefore the extremal problem (1.2) has solution if $i \in (-\infty,0)\cup(n,+\infty)$ and it must be replaced by the extremal problem

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ)\colon T \in GL(n)\}, \tag{1.3}$$

for $i \in (0,n)$. In section 2, we characterize the solutions of (1.2) in terms of covariance matrices of some probabilities whenever $i \in (-\infty,0)\cup[n+1,+\infty)$. We will use essentially the same notation that appears in [Ga] and [Sc].

2. Main results

If $K \subseteq \mathbb{R}^n$ is a convex body and $i \in \mathbb{R}$, we study the extremal values of $W_i(TK)\tilde{W}_i(TK)^\circ$, where T runs over all regular transformation $T \in GL(n)$. As we noticed in the introduction, we can verify for necessary and sufficient conditions for a convex body K and $i \in \mathbb{R}$

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\},$$

if $i \in (-\infty, 0) \cup (n, +\infty)$ or

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\},$$

if $i \in (0, n)$. The following result gives a characterization of the solution of this problem when $i \in (-\infty, 0) \cup [n+1, +\infty)$.

THEOREM 2.1: *Let $i \in (-\infty, 0) \cup [n+1, \infty)$, $n \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body (i.e., $h_K(\cdot)$ and $h_{K^\circ}(\cdot)$ are twice continuously differentiable) having the origin in its interior. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\}$.

(ii) *For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$*

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), T^\star u\rangle d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle d\sigma(u).$$

(iii) *For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ symmetric*

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle d\sigma(u).$$

(iv) *For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$*

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle d\sigma(u).$$

(v) *For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ symmetric*

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle d\sigma(u).$$

(vi) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ): T \in GL(n)\}$ *and the minimum on* $SL(n)$ *is unique up to orthogonal transformation.*

Proof: (i) $\Longrightarrow$ (ii). For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we can define $T_\varepsilon = I_n + \varepsilon T \in GL(n)$. Since for every $\varepsilon ||T|| < \frac{1}{2}$

$$(I_n + \varepsilon T)^{-1}u = u - \varepsilon Tu + O(\varepsilon^2),$$
$$\rho_K((I_n + \varepsilon T)^{-1}u) = \frac{1}{h_{K^\circ}(u) - \varepsilon\langle \nabla h_{K^\circ}(u), Tu\rangle + O(\varepsilon^2)},$$

we get that

$$\begin{aligned}
&\tilde{W}_i(T_\varepsilon K)\\
&\quad = \tilde{W}_i(K) - \frac{i-n}{n}\varepsilon \int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle Tu, \nabla h_{K^\circ}(u)\rangle d\sigma(u) + O(\varepsilon^2),\\
&\tilde{W}_i((T_\varepsilon K)^\circ)\\
&\quad = \tilde{W}_i(K^\circ) + \frac{i-n}{n}\varepsilon \int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle T^\star u, \nabla h_K(u)\rangle d\sigma(u) + O(\varepsilon^2).
\end{aligned}$$

By hypothesis $\tilde{W}_i(K)\tilde{W}_i(K^\circ) \le \tilde{W}_i(T_\varepsilon K)\tilde{W}_i((T_\varepsilon K)^\circ)$, therefore if we let $\varepsilon \longrightarrow 0^+$, by using the last expressions for $\tilde{W}_i(T_\varepsilon K)$ and $\tilde{W}_i((T_\varepsilon K)^\circ)$ we get that for every $T \in GL(n)$

$$\begin{aligned}
&\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle d\sigma(u)\\
&\qquad\qquad \le \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), T^\star u\rangle d\sigma(u),
\end{aligned}$$

but if we replace T by $-T$ in the last expression we obtain (ii).

(ii) $\Longrightarrow$ (iii) and (iv) $\Longleftrightarrow$ (v) are trivial.

In order to prove (iii) $\Longleftrightarrow$ (iv) it is enough to check that the following assertions are equivalent:

(iii′) For every $\theta \in S^{n-1}$

$$\begin{aligned}
&\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), \theta\rangle\langle u, \theta\rangle d\sigma(u)\\
&\qquad\qquad = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), \theta\rangle\langle u, \theta\rangle d\sigma(u).
\end{aligned}$$

(iv′) For every $\theta \in S^{n-1}$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, \theta\rangle^2 d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, \theta\rangle^2 d\sigma(u).$$

It was proved in [BR] (by using Laplace–Beltrami operator techniques) that for every *"smooth enough"* convex bodies $L, M \subseteq \mathbb{R}^n$ with 0 in their interior

$$\begin{aligned}(n-i)\int_{S^{n-1}} \rho_L^{n-i+1}(u)\rho_M^i(u)\langle\nabla h_{L^\circ}(u),\theta\rangle\langle u,\theta\rangle d\sigma(u) \\ = \int_{S^{n-1}} \rho_L^{n-i}(u)\rho_M^i(u)d\sigma(u) \\ - i\int_{S^{n-1}} \rho_L^{n-i}(u)\rho_M^{i+1}(u)\langle\nabla h_{M^\circ}(u),\theta\rangle\langle u,\theta\rangle d\sigma(u),\end{aligned}$$

for all $\theta \in S^{n-1}$. Therefore, if we put $L = K$, $M = D_n$ and $L = K^\circ$, $M = D_n$ in the last expression we obtain that (iv′) $\Longleftrightarrow$ (iii′).

The final part of the proof of the theorem is different, depending on the index i, and we prove (v) $\Rightarrow$ (vi) for $i < 0$ and (iii) $\Rightarrow$ (vi) for $i \geq n+1$.

(v) $\Rightarrow$ (vi) (for $i < 0$). It is easy to check that we only have to consider diagonal operators $T \in SL(n)$ with diagonal elements $d_1, \dots, d_n > 0$.

If we consider the case $i \leq -1$, since $T \in SL(n)$ it can be easily checked that

$$\tilde{W}_i(TK) = \frac{n-i}{n}\int_K \frac{dx}{|Tx|^i} = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\,|Tu|^{-i}\,d\sigma(u).$$

Hence, by using Hölder's inequality in the last expression, it follows that

$$\tilde{W}_i(TK) \geq \tilde{W}_i(K)^{i+1}\Big(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\,|Tu|\,d\sigma(u)\Big)^{-i}$$

and, since $0 \leq \langle u, Tu\rangle \leq |Tu|$, we get that

$$\tilde{W}_i(TK) \geq \tilde{W}_i(K)\Big(\frac{\tilde{W}_i(K)}{\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u,Tu\rangle d\sigma(u)}\Big)^i.$$

If we use the same philosophy with $\tilde{W}_i(TK)^\circ$, we obtain that

$$\begin{aligned}&\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)\cdot \\ &\cdot\Big(\frac{\tilde{W}_i(K)\tilde{W}_i(K^\circ)}{\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u,Tu\rangle d\sigma(u)\frac{1}{n}\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u,T^{-1}u\rangle d\sigma(u)}\Big)^i.\end{aligned}$$

By using the hypothesis, we get that

$$\frac{\tilde{W}_i(K)}{\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u,T^{-1}u\rangle d\sigma(u)} = \frac{\tilde{W}_i(K^\circ)}{\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u,T^{-1}u\rangle d\sigma(u)},$$

hence, since $i < 0$, it is enough to prove that

$$\text{(2.4)}\quad \tilde{W}_i(K)^2 \leq \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u,Tu\rangle d\sigma(u)\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u,T^{-1}u\rangle d\sigma(u).$$

For every $u \in S^{n-1}$

$$\begin{aligned}\langle u, Tu\rangle\langle u, T^{-1}u\rangle =&\left(\sum_{j=1}^{n} d_j u_j^2\right)\left(\sum_{j=1}^{n} d_j^{-1} u_j^2\right)\\ \geq&\left(\prod_{j=1}^{n} d_j^{u_j^2}\right)\left(\prod_{j=1}^{n} d_j^{-u_j^2}\right) = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{W}_i(K) \leq&\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)(\langle u, Tu\rangle)^{1/2}(\langle u, T^{-1}u\rangle)^{1/2} d\sigma(u)\\ \leq&\left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle d\sigma(u)\right)^{1/2}\\ &\cdot\left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, T^{-1}u\rangle d\sigma(u)\right)^{1/2},\end{aligned}$$

which shows that $\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)$.

Now, if $-1 < i < 0$ and we take a diagonal operator $T \in SL(n)$ with diagonal elements $d_1, \dots, d_n > 0$, since $f(x) = x^{-i/2}$ is concave in $[0, +\infty)$ we get that

$$\begin{aligned}\tilde{W}_i(TK) =&\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\,|Tu|^{-i}\, d\sigma(u)\\ =&\frac{1}{n}\int_{S^{n-1}} \left(\sum_{j=1}^{n} d_j^2 u_j^2\right)^{-i/2} \rho_K^{n-i}(u) d\sigma(u)\\ \geq&\frac{1}{n}\int_{S^{n-1}} \sum_{j=1}^{n} d_j^{-i} u_j^2 \rho_K^{n-i}(u) d\sigma(u)\\ \geq&\frac{1}{n}\int_{S^{n-1}} \prod_{j=1}^{n} d_j^{-iu_j^2} \rho_K^{n-i}(u) d\sigma(u).\end{aligned}$$

On the other hand, by hypothesis we can ensure that

$$\begin{aligned}\tilde{W}_i((TK)^\circ) =&\frac{1}{n}\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\,\left|T^{-1}u\right|^{-i}\, d\sigma(u)\\ \geq&\frac{1}{n}\int_{S^{n-1}} \sum_{j=1}^{n} d_j^{i} u_j^2 \rho_{K^\circ}^{n-i}(u) d\sigma(u)\\ =&\frac{\tilde{W}_i(K^\circ)}{\tilde{W}_i(K)}\frac{1}{n}\int_{S^{n-1}} \sum_{j=1}^{n} d_j^{i} u_j^2 \rho_K^{n-i}(u) d\sigma(u)\\ \geq&\frac{\tilde{W}_i(K^\circ)}{\tilde{W}_i(K)}\frac{1}{n}\int_{S^{n-1}} \prod_{j=1}^{n} d_j^{iu_j^2} \rho_K^{n-i}(u) d\sigma(u).\end{aligned}$$

Now, by combining the last two expressions and by using Cauchy–Schwartz inequality we obtain that

$$\begin{aligned}
\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ)\frac{\tilde{W}_i(K)}{\tilde{W}_i(K^\circ)} \\
&\geq \left(\frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^n d_j^{-iu_j^2/2}\prod_{j=1}^n d_j^{iu_j^2/2}\rho_K^{n-i}(u)d\sigma(u)\right)^2 \\
&= \tilde{W}_i(K)^2,
\end{aligned}$$

which proves that $\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)$.

The uniqueness of the extremal position on $SL(n)$ up to orthogonal transformation is a straightforward consequence of the fact that the equality in the AGM-inequality only happens for $d_1 = \cdots = d_n$, which means that $T = I_n$.

(iii) $\Longrightarrow$ (vi) ($i \geq n+1$). If $T \in GL(n)$, there exist orthogonal transformations $U, V \in O(n)$ and diagonal transformation $T \in GL(n)$ such that $T = VDU$. It is easy to check that if $K_1 = UK$, then $\tilde{W}_i(K) = \tilde{W}_i(K_1)$, $W_i(K^\circ) = \tilde{W}_i(K_1^\circ)$, and if K verifies (iii) then K_1 also verifies (iii).

If $i = n+1$, by hypothesis we can choose $V_1 \in O(n)$ and diagonal transformation D_1 with diagonal elements $d_1, \ldots, d_n$ such that $T = V_1 D_1 U$ and for every $j = 1, \ldots, n$

$$d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u) \geq 0.$$

Now, by using the hypothesis for K_1 and this decomposition of T we get that

$$\begin{aligned}
\tilde{W}_{n+1}(TK) &= \tilde{W}_{n+1}(D_1K_1) \\
&= \frac{1}{n}\int_{S^{n-1}} h_{K_1^\circ}(D_1^{-1}u)d\sigma(u) \\
&\geq \frac{1}{n}\int_{S^{n-1}} \langle \nabla h_{K_1^\circ}(u), D_1^{-1}u\rangle d\sigma(u) \\
&= \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_{K_1}(u), D_1^{-1}u\rangle d\sigma(u) \\
&= \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\frac{1}{n}\sum_{j=1}^n d_j^{-1}\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u) \\
&\geq 0.
\end{aligned}$$

In the same way

$$\begin{aligned}\tilde{W}_{n+1}((TK)^\circ) =&\tilde{W}_{n+1}(D_1^{-1}K_1)^\circ\\ =&\frac{1}{n}\int_{S^{n-1}} h_{K_1}(D_1u)d\sigma(u)\\ \geq&\frac{1}{n}\int_{S^{n-1}}\langle\nabla h_{K_1}(u), D_1u\rangle d\sigma(u)\\ =&\frac{1}{n}\sum_{j=1}^n d_j\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u)\geq 0.\end{aligned}$$

Therefore, by combining these two expressions and Cauchy–Schwarz inequality we get that

$$\begin{aligned}&\tilde{W}_{n+1}(TK)\tilde{W}_{n+1}((TK)^\circ)\\ &\quad\geq\frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\left(\frac{1}{n}\sum_{j=1}^n d_j^{-1}\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u)\right)\\ &\quad\cdot\left(\frac{1}{n}\sum_{j=1}^n d_j\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u)\right)\\ &\quad\geq\frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\left(\frac{1}{n}\sum_{j=1}^n |d_j|^{1/2}|d_j^{-1}|^{1/2}\left|\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)d\sigma(u)\right|\right)^2\\ &\quad\geq\frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\left(\frac{1}{n}\int_{S^{n-1}}\langle\nabla h_{K_1}(u),u\rangle d\sigma(u)\right)^2.\end{aligned}$$

Now, by the homogeneity of $h_{K_1}(\cdot)$ we obtain that

$$\begin{aligned}\tilde{W}_{n+1}(TK)\tilde{W}_{n+1}((TK)^\circ)\geq&\frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)}\left(\frac{1}{n}\int_{S^{n-1}} h_{K_1}(u)d\sigma(u)\right)^2\\ =&\tilde{W}_{n+1}(K_1)\tilde{W}_{n+1}(K_1^\circ)=\tilde{W}_{n+1}(K)\tilde{W}_{n+1}(K^\circ).\end{aligned}$$

If $i > n+1$ the proof can be completed by using the same ideas, since for every $T\in GL(n)$ symmetric we can find $V_1, U\in O(n)$ and diagonal transformation D_1 with diagonal elements $d_1,\ldots,d_n$ such that $T = V_1D_1U$ and for every $j = 1,\ldots,n$

$$d_j\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\geq 0,$$

where $K_1 = UK$. Hence, by using Hölder inequality ($p = i-n$, $q = \frac{i-n}{i-n-1}$) we get that

$$\begin{aligned}\tilde{W}_i(TK) =&\tilde{W}_i(D_1K_1)\\ \geq&\tilde{W}_i(K_1)^{n-i+1}\left(\frac{1}{n}\int_{S^{n-1}} h_{(D_1K_1)^\circ}(u)\rho_{K_1}^{n-i+1}(u)d\sigma(u)\right)^{i-n}.\end{aligned}$$

But, by hypothesis

$$\begin{aligned}\int_{S^{n-1}} & h_{(D_1K_1)^\circ}(u)\rho_{K_1}^{n-i+1}(u)d\sigma(u)\\ &\geq \int_{S^{n-1}}\langle \nabla h_{K_1^\circ}(u), D_1^{-1}u\rangle \rho_{K_1}^{n-i+1}(u)d\sigma(u)\\ &= \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)}\int_{S^{n-1}}\langle \nabla h_{K_1}(u), D_1^{-1}u\rangle \rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\\ &= \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)}\sum_{j=1}^n d_j^{-1}\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\geq 0,\end{aligned}$$

hence

$$\tilde{W}_i(TK)\geq \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)^{i-n}}\left(\frac{1}{n}\sum_{j=1}^n d_j^{-1}\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\right)^{i-n}.$$

If we use the same technique with $\tilde{W}_i(TK)^\circ$ and we combine both expressions we get that

$$\begin{aligned}&\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ)\\ &\geq\frac{\tilde{W}_i(K_1)}{(\tilde{W}_i(K_1^\circ))^{2(i-n)-1}}\left(\frac{1}{n}\sum_{j=1}^n d_j^{-1}\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\right)^{i-n}\\ &\quad\cdot\left(\frac{1}{n}\sum_{j=1}^n d_j\int_{S^{n-1}} u_j\frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)d\sigma(u)\right)^{i-n}.\end{aligned}$$

Finally, by using again the Cauchy–Schwarz inequality we conclude the result. The uniqueness of the solution up to orthogonal transformations for $T\in SL(n)$ is straightforward, since the equality in the Cauchy–Schwarz inequality implies in the last expression that $|d_1|=\cdots=|d_n|$ and hence $T=I_n$. ∎

COROLLARY 2.2: *Let $K\subseteq\mathbb{R}^n$ be a "smooth enough" convex body having the origin in its interior. Then the following assertions are equivalent:*

(i) $M(K)M^\star(K)=\min\{M(TK)M^\star(TK)\colon T\in GL(n)\}$.

(ii) *For every $T\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$*

$$\begin{aligned}M^\star(K)&\int_{S^{n-1}}\langle\nabla h_{K^\circ}(u),T^\star u\rangle d\sigma(u)\\ &= M(K)\int_{S^{n-1}}\langle\nabla h_K(u),Tu\rangle d\sigma(u).\end{aligned}$$

(iii) *For every* $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$

$$M^\star(K) \int_{S^{n-1}} \|u\|_K \langle u, Tu\rangle d\sigma(u) = M(K) \int_{S^{n-1}} \|u\|_{K^\circ} \langle u, Tu\rangle d\sigma(u).$$

(iv) $M(K)M^\star(K) = \min\{M(TK)M^\star(TK) \colon T \in GL(n)\}$ *and the minimum is unique on* $SL(n)$ *up to orthogonal transformation.*

Remark 2.3: If $i \in [0, n+1)$ the assertions (ii) and (iv) (and therefore (iii) and (v)) in Theorem 2.1 are necessary conditions for a convex body $K \subseteq \mathbb{R}^n$ *"smooth enough"* and such that the origin is in its interior to verify that

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\}$$

if $i \in (0, n)$ or

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\}$$

if $i \in (n, n+1)$. In both cases it can be proved that (iii) and (iv) are equivalent conditions.

Remark 2.4: If $K \subseteq \mathbb{R}^n$ is a centrally symmetric convex body then the last corollary implies Theorem 1.1, since the probability μ_K given by

$$d\mu_K(u) = \frac{\|u\|_K}{\int_{S^{n-1}} \|v\|_K d\sigma(v)} d\sigma(u)$$

has mean 0.

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